Drinfeld twist and symmetric Bethe vectors of the open XYZ chain with non-diagonal boundary terms

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Abstract

With the help of the Drinfeld twist or factorizing F-matrix for the eight-vertex solid-on-solid (SOS) model, we find that in the F-basis provided by the twist the two sets of pseudo-particle creation operators simultaneously take completely symmetric and polarization free form. This allows us to obtain the explicit and completely symmetric expressions of the two sets of Bethe states of the model.

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1 Introduction

The algebraic Bethe ansatz [1] has been proven to provide a powerful tool of solving eigenvalue problems of quantum integrable systems such as quantum spin chains. In this framework, the pseudo-particle creation and annihilation operators are constructed by off-diagonal matrix elements of the so-called monodromy matrix. The Bethe states (eigenstates) of transfer matrix are obtained by applying creation operators to the reference state (or pseudo-vacuum state). However, the apparently simple action of creation operators is plagued with non-local effects arising from polarization clouds or compensating exchange terms on the level of local operators. This make the explicit construction of the Bethe states challenging.

Progress for obtaining explicit expressions of the Bethe states has been made for the XXX and XXZ spin chains with periodic boundary conditions (or the closed XXX and XXZ chains) [2] by using the so-called F-basis provided by the Drinfeld twist or factorizing F-matrix [3]. In the F-basis, the pseudo-particle creation and annihilation operators of the models take completely symmetric forms and contain no compensating exchange terms on the level of local operators (i.e. polarization free). As a result, the Bethe states of the models are simplified dramatically and can be written down explicitly [4]. Similar results have been obtained for other models with periodic boundary conditions [5, 6, 7, 8, 9].

It was shown [10, 11] that the F-matrices of the closed XXX and XXZ chains also make the pseudo-particle creation operators of the open XXX and XXZ chains with diagonal boundary terms [12] polarization free. This is mainly due to the fact that the closed chain and the corresponding open chain with diagonal boundary terms share the same reference state. However, the story for the open XXZ chain with non-diagonal boundary terms is quite different [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]. Firstly, the reference state (all spin up state) of the closed chain is no longer a reference state of the open chain with non-diagonal boundary terms [14, 15, 16]. Secondly, at least two reference states (and thus two sets of Bethe states) are needed [30] for the open XXZ chain with non-diagonal boundary terms in order to obtain its complete spectrum [31, 24]. As a consequence, the F-matrix found in [2] is no longer the *desirable* F-matrix for the open XXZ chain with non-diagonal boundary terms. Recently, we have succeeded in obtaining the factorizing F-matrices for the open XXZ chain with a non-diagonal boundary terms [32] and using the F-matrices to construct the determinant representations of the DW partition function of the six-vertex

model with a non-diagonal reflection end [33, 34] and the scalar products of the Bethe states of the open XXZ chain [35].

In this paper, we focus on the most general Heisenberg spin chain—the open XYZ chain [36, 37] with a non-diagonal boundary terms whose trigonometric/rational limit gives the open XXZ/XXX chain. With the help of the F-matrices of the eight-vertex SOS model [6], we find that in the F-basis the two sets of pseudo-particle creation operators (acting on the two reference states) of the boundary model simultaneously take completely symmetric and polarization free forms. These enable us to derive the explicit and completely symmetric expressions of the two sets of Bethe states of the model. Moreover, the coefficients in these expressions can be expressed in terms of a single determinant. Such a single determinant representation will be essential for the study of the scalar products of the Bethe vectors of the open XYZ chain with non-diagonal boundary terms.

The paper is organized as follows. In section 2, we briefly describe the open XYZ chain with non-diagonal boundary terms and introduce the pseudo-particle creation operators and the two sets of Bethe states of the model. In section 3, we introduce the face picture of the model and express the two sets of Bethe states in terms of their face-picture versions. In section 4, with the help of the F-matrix of the eight-vertex SOS model, we obtain the completely symmetric and polarization free representations of the pseudo-particle creation operators. In section 5, we give the complete symmetric expressions of the two sets of Bethe states in the F-basis, in which the coefficients can be expressed in terms of a single determinant respectively. In section 6, we summarize our results and give some discussions.

2 The inhomogeneous spin-$\frac{1}{2}$ XYZ open chain

Let us fix τ such that $\mathrm{Im}(\tau) > 0$ and a generic complex number η. Introduce the following elliptic functions

$$\theta\left[\begin{array}{c} a \\ b \end{array}\right](u,\tau) = \sum_{n=-\infty}^{\infty} \exp\left\{i\pi\left[(n+a)^2\tau + 2(n+a)(u+b)\right]\right\}, \tag{2.1}$$

$$\theta^{(j)}(u) = \theta\left[\begin{array}{c} \frac{1}{2}-\frac{j}{2} \\ \frac{1}{2} \end{array}\right](u,2\tau), \quad j=1,2; \qquad \sigma(u) = \theta\left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array}\right](u,\tau). \tag{2.2}$$

The σ-function[1] satisfies the so-called Riemann identity:

$$
\begin{aligned}
&\sigma(u+x)\sigma(u-x)\sigma(v+y)\sigma(v-y)-\sigma(u+y)\sigma(u-y)\sigma(v+x)\sigma(v-x) \\
&\quad =\sigma(u+v)\sigma(u-v)\sigma(x+y)\sigma(x-y),
\end{aligned} \tag{2.3}
$$

which will be useful in the following. Moreover, for any $\alpha=(\alpha_1,\alpha_2)$, $\alpha_1,\alpha_2\in\mathbb{Z}_2$, we can introduce a function $\sigma_\alpha(u)$ as follow

$$
\sigma_\alpha(u)=\theta\left[\begin{array}{c}\frac{1}{2}+\frac{\alpha_1}{2}\\ \frac{1}{2}+\frac{\alpha_2}{2}\end{array}\right](u,\tau),\qquad \alpha_1,\alpha_2\in\mathbb{Z}_2. \tag{2.4}
$$

The above definition implies the identification $\sigma_{(0,0)}(u)=\sigma(u)$.

Let V be a two-dimensional vector space $\mathbb{C}^2$ and $\{\epsilon_i|i=1,2\}$ be the orthonormal basis of V such that $\langle\epsilon_i,\epsilon_j\rangle=\delta_{ij}$. The well-known eight-vertex model R-matrix $\overline{R}(u)\in\mathrm{End}(V\otimes V)$ is given by

$$
\overline{R}(u)=\begin{pmatrix} a(u) & & & d(u)\\ & b(u) & c(u) & \\ & c(u) & b(u) & \\ d(u) & & & a(u)\end{pmatrix}. \tag{2.5}
$$

The non-vanishing matrix elements are [37]

$$
\begin{aligned}
a(u)&=\frac{\theta^{(1)}(u)\,\theta^{(0)}(u+\eta)\,\sigma(\eta)}{\theta^{(1)}(0)\,\theta^{(0)}(\eta)\,\sigma(u+\eta)}, &\quad b(u)&=\frac{\theta^{(0)}(u)\,\theta^{(1)}(u+\eta)\,\sigma(\eta)}{\theta^{(1)}(0)\,\theta^{(0)}(\eta)\,\sigma(u+\eta)},\\
c(u)&=\frac{\theta^{(1)}(u)\,\theta^{(1)}(u+\eta)\,\sigma(\eta)}{\theta^{(1)}(0)\,\theta^{(1)}(\eta)\,\sigma(u+\eta)}, &\quad d(u)&=\frac{\theta^{(0)}(u)\,\theta^{(0)}(u+\eta)\,\sigma(\eta)}{\theta^{(1)}(0)\theta^{(1)}(\eta)\,\sigma(u+\eta)}.
\end{aligned} \tag{2.6}
$$

Here u is the spectral parameter and η is the so-called crossing parameter. The R-matrix satisfies the quantum Yang-Baxter equation (QYBE)

$$
\overline{R}_{1,2}(u_1-u_2)\overline{R}_{1,3}(u_1-u_3)\overline{R}_{2,3}(u_2-u_3)=\overline{R}_{2,3}(u_2-u_3)\overline{R}_{1,3}(u_1-u_3)\overline{R}_{1,2}(u_1-u_2). \tag{2.7}
$$

Throughout we adopt the standard notation: for any matrix $A\in\mathrm{End}(V)$, A_j (or A^j)is an embedding operator in the tensor space $V\otimes V\otimes\cdots$, which acts as A on the j-th space and as identity on the other factor spaces; $R_{i,j}(u)$ is an embedding operator of R-matrix in the tensor space, which acts as identity on the factor spaces except for the i-th and j-th ones.

[1]Our σ-function is the ϑ-function $\vartheta_1(u)$ [38]. It has the following relation with the *Weierstrassian* σ-function $\sigma_w(u)$: $\sigma_w(u)\propto e^{\eta_1u^2}\sigma(u)$ with $\eta_1=\pi^2(\frac{1}{6}-4\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}})$ and $q=e^{i\tau}$.

One introduces the "row-to-row" (or one-row) monodromy matrix $T(u)$, which is an 2×2 matrix with elements being operators acting on $V^{\otimes N}$, where $N = 2M$ (M being a positive integer),

$$T_0(u) = \overline{R}_{0,N}(u - z_N)\overline{R}_{0,N-1}(u - z_{N-1}) \cdots \overline{R}_{0,1}(u - z_1). \tag{2.8}$$

Here $\{z_j | j = 1, \cdots, N\}$ are arbitrary free complex parameters which are usually called inhomogeneous parameters.

Integrable open chain can be constructed as follows [12]. Let us introduce a pair of K-matrices $K^-(u)$ and $K^+(u)$. The former satisfies the reflection equation (RE)

$$\begin{aligned} &\overline{R}_{1,2}(u_1 - u_2)K_1^-(u_1)\overline{R}_{2,1}(u_1 + u_2)K_2^-(u_2) \\ &\quad = K_2^-(u_2)\overline{R}_{1,2}(u_1 + u_2)K_1^-(u_1)\overline{R}_{2,1}(u_1 - u_2), \end{aligned} \tag{2.9}$$

and the latter satisfies the dual RE

$$\begin{aligned} &\overline{R}_{1,2}(u_2 - u_1)K_1^+(u_1)\overline{R}_{2,1}(-u_1 - u_2 - 2\eta)K_2^+(u_2) \\ &\quad = K_2^+(u_2)\overline{R}_{1,2}(-u_1 - u_2 - 2\eta)K_1^+(u_1)\overline{R}_{2,1}(u_2 - u_1). \end{aligned} \tag{2.10}$$

For open spin-chains, instead of the standard "row-to-row" monodromy matrix $T(u)$ (2.8), one needs to consider the "double-row" monodromy matrix $\mathbb{T}(u)$

$$\mathbb{T}(u) = T(u)K^-(u)\hat{T}(u), \quad \hat{T}(u) = T^{-1}(-u). \tag{2.11}$$

Then the double-row transfer matrix of the XYZ chain with open boundary (or the open XYZ chain) is given by

$$t(u) = tr(K^+(u)\mathbb{T}(u)). \tag{2.12}$$

The QYBE and (dual) REs lead to that the transfer matrices with different spectral parameters commute with each other [12]: $[t(u), t(v)] = 0$. This ensures the integrability of the open XYZ chain.

In this paper, we consider the K-matrix $K^-(u)$ which is a generic solution [39, 40] to the RE (2.9) associated with the R-matrix (2.5)

$$K^-(u) = k_0^-(u) + k_x^-(u)\,\sigma^x + k_y^-\,\sigma^y + k_z^-(u)\,\sigma^z, \tag{2.13}$$

where $\sigma^x, \sigma^y, \sigma^z$ are the Pauli matrices and the coefficient functions are

$$k_0^-(u) = \frac{\sigma(2u)\,\sigma(\lambda_1+\lambda_2-\frac{1}{2})\,\sigma(\lambda_1+\xi)\,\sigma(\lambda_2+\xi)}{2\,\sigma(u)\,\sigma(-u+\lambda_1+\lambda_2-\frac{1}{2})\,\sigma(\lambda_1+\xi+u)\,\sigma(\lambda_2+\xi+u)},$$

$$k_x^-(u) = \frac{\sigma(2u)\,\sigma_{(1,0)}(\lambda_1+\lambda_2-\frac{1}{2})\,\sigma_{(1,0)}(\lambda_1+\xi)\,\sigma_{(1,0)}(\lambda_2+\xi)}{2\,\sigma_{(1,0)}(u)\,\sigma(-u+\lambda_1+\lambda_2-\frac{1}{2})\,\sigma(\lambda_1+\xi+u)\,\sigma(\lambda_2+\xi+u)},$$

$$k_y^-(u) = \frac{i\,\sigma(2u)\,\sigma_{(1,1)}(\lambda_1+\lambda_2-\frac{1}{2})\,\sigma_{(1,1)}(\lambda_1+\xi)\,\sigma_{(1,1)}(\lambda_2+\xi)}{2\,\sigma_{(1,1)}(u)\,\sigma(-u+\lambda_1+\lambda_2-\frac{1}{2})\,\sigma(\lambda_1+\xi+u)\,\sigma(\lambda_2+\xi+u)},$$

$$k_z^-(u) = \frac{\sigma(2u)\,\sigma_{(0,1)}(\lambda_1+\lambda_2-\frac{1}{2})\,\sigma_{(0,1)}(\lambda_1+\xi)\,\sigma_{(0,1)}(\lambda_2+\xi)}{2\,\sigma_{(0,1)}(u)\,\sigma(-u+\lambda_1+\lambda_2-\frac{1}{2})\,\sigma(\lambda_1+\xi+u)\,\sigma(\lambda_2+\xi+u)}. \tag{2.14}$$

At the same time, we introduce the corresponding *dual* K-matrix $K^+(u)$ which is a generic solution to the dual reflection equation (2.10) with a particular choice of the free boundary parameters according to those of $K^-(u)$ (2.13)-(2.14):

$$K^+(u) = k_0^+(u) + k_x^+(u)\,\sigma^x + k_y^+\,\sigma^y + k_z^+(u)\,\sigma^z, \tag{2.15}$$

with the coefficient functions

$$k_0^+(u) = \frac{\sigma(-2u-2\eta)\sigma(\lambda_1+\lambda_2+\eta-\frac{1}{2})\sigma(\lambda_1+\bar{\xi})\sigma(\lambda_2+\bar{\xi})}{2\sigma(-u-\eta)\sigma(u+\eta+\lambda_1+\lambda_2-\frac{1}{2})\sigma(\lambda_1+\bar{\xi}-u-\eta)\sigma(\lambda_2+\bar{\xi}-u-\eta)},$$

$$k_x^+(u) = \frac{\sigma(-2u-2\eta)\sigma_{(1,0)}(\lambda_1+\lambda_2+\eta-\frac{1}{2})\sigma_{(1,0)}(\lambda_1+\bar{\xi})\sigma_{(1,0)}(\lambda_2+\bar{\xi})}{2\sigma_{(1,0)}(-u-\eta)\sigma(u+\eta+\lambda_1+\lambda_2-\frac{1}{2})\sigma(\lambda_1+\bar{\xi}-u-\eta)\sigma(\lambda_2+\bar{\xi}-u-\eta)},$$

$$k_y^+(u) = \frac{i\,\sigma(-2u-2\eta)\sigma_{(1,1)}(\lambda_1+\lambda_2+\eta-\frac{1}{2})\sigma_{(1,1)}(\lambda_1+\bar{\xi})\sigma_{(1,1)}(\lambda_2+\bar{\xi})}{2\sigma_{(1,1)}(-u-\eta)\sigma(u+\eta+\lambda_1+\lambda_2-\frac{1}{2})\sigma(\lambda_1+\bar{\xi}-u-\eta)\sigma(\lambda_2+\bar{\xi}-u-\eta)},$$

$$k_z^+(u) = \frac{\sigma(-2u-2\eta)\sigma_{(0,1)}(\lambda_1+\lambda_2+\eta-\frac{1}{2})\sigma_{(0,1)}(\lambda_1+\bar{\xi})\sigma_{(0,1)}(\lambda_2+\bar{\xi})}{2\sigma_{(0,1)}(-u-\eta)\sigma(u+\eta+\lambda_1+\lambda_2-\frac{1}{2})\sigma(\lambda_1+\bar{\xi}-u-\eta)\sigma(\lambda_2+\bar{\xi}-u-\eta)}. \tag{2.16}$$

The K-matrices $K^{\mp}(u)$ depend on four free boundary parameters $\{\lambda_1,\,\lambda_2,\,\xi,\,\bar{\xi}\}$. It is very convenient to introduce a vector $\lambda \in V$ associated with the boundary parameters $\{\lambda_i\}$,

$$\lambda = \sum_{k=1}^{2} \lambda_k \epsilon_k. \tag{2.17}$$

2.1 Vertex-face correspondence

Let us briefly review the face-type R-matrix associated with the six-vertex model.

Set

$$\hat{\imath} = \epsilon_i - \bar{\epsilon}, \quad \bar{\epsilon} = \frac{1}{2}\sum_{k=1}^{2}\epsilon_k, \quad i = 1, 2, \qquad \text{then} \sum_{i=1}^{2}\hat{\imath} = 0. \tag{2.18}$$

Let $\mathfrak{h}$ be the Cartan subalgebra of A_1 and $\mathfrak{h}^*$ be its dual. A finite dimensional diagonalizable $\mathfrak{h}$-module is a complex finite dimensional vector space W with a weight decomposition $W = \oplus_{\mu\in\mathfrak{h}^*}W[\mu]$, so that $\mathfrak{h}$ acts on $W[\mu]$ by $x\,v = \mu(x)\,v$, $(x \in \mathfrak{h},\ v \in W[\mu])$. For example, the non-zero weight spaces of the fundamental representation $V_{\Lambda_1} = \mathbb{C}^2 = V$ are

$$W[\hat{\imath}] = \mathbb{C}\epsilon_i, \ \ i = 1, 2. \tag{2.19}$$

For a generic $m \in V$, define

$$m_i = \langle m, \epsilon_i\rangle, \quad m_{ij} = m_i - m_j = \langle m, \epsilon_i - \epsilon_j\rangle, \quad i, j = 1, 2. \tag{2.20}$$

Let $R(u, m) \in \mathrm{End}(V \otimes V)$ be the R-matrix of the eight-vertex SOS model [37] given by

$$R(u; m) = \sum_{i=1}^{2} R(u; m)^{ii}_{ii} E_{ii}\otimes E_{ii} + \sum_{i\neq j}^{2}\left\{R(u; m)^{ij}_{ij} E_{ii}\otimes E_{jj} + R(u; m)^{ji}_{ij} E_{ji}\otimes E_{ij}\right\}, \tag{2.21}$$

where E_{ij} is the matrix with elements $(E_{ij})^l_k = \delta_{jk}\delta_{il}$. The coefficient functions are

$$R(u; m)^{ii}_{ii} = 1, \quad R(u; m)^{ij}_{ij} = \frac{\sigma(u)\sigma(m_{ij} - \eta)}{\sigma(u + \eta)\sigma(m_{ij})}, \quad i \neq j, \tag{2.22}$$

$$R(u; m)^{ji}_{ij} = \frac{\sigma(\eta)\sigma(u + m_{ij})}{\sigma(u + \eta)\sigma(m_{ij})}, \quad i \neq j, \tag{2.23}$$

and m_{ij} is defined in (2.20). The R-matrix satisfies the dynamical (modified) quantum Yang-Baxter equation (or the star-triangle relation) [37]

$$\begin{aligned} &R_{1,2}(u_1 - u_2; m - \eta h^{(3)})R_{1,3}(u_1 - u_3; m)R_{2,3}(u_2 - u_3; m - \eta h^{(1)}) \\ &\quad = R_{2,3}(u_2 - u_3; m)R_{1,3}(u_1 - u_3; m - \eta h^{(2)})R_{1,2}(u_1 - u_2; m). \end{aligned} \tag{2.24}$$

Here we have adopted

$$R_{1,2}(u, m - \eta h^{(3)})\, v_1 \otimes v_2 \otimes v_3 = (R(u, m - \eta\mu) \otimes \mathrm{id})\, v_1 \otimes v_2 \otimes v_3, \quad \text{if } v_3 \in W[\mu]. \tag{2.25}$$

Moreover, one may check that the R-matrix satisfies the weight conservation condition,

$$\left[h^{(1)} + h^{(2)},\ R_{1,2}(u; m)\right] = 0, \tag{2.26}$$

the unitary condition,

$$R_{1,2}(u;m)\,R_{2,1}(-u;m)=\mathrm{id}\otimes\mathrm{id}, \tag{2.27}$$

and the crossing relation

$$R(u;m)^{kl}_{ij}=\varepsilon_l\,\varepsilon_j\frac{\sigma(u)\sigma((m-\eta\hat{\imath})_{21})}{\sigma(u+\eta)\sigma(m_{21})}R(-u-\eta;m-\eta\hat{\imath})^{\bar{j}\,k}_{\bar{l}\,i}, \tag{2.28}$$

where

$$\varepsilon_1=1,\,\varepsilon_2=-1,\quad \text{and } \bar{1}=2,\,\bar{2}=1. \tag{2.29}$$

Let us introduce two intertwiners which are 2-component column vectors $\phi_{m,m-\eta\hat{\jmath}}(u)$ labelled by $\hat{1}$, $\hat{2}$. The k-th element of $\phi_{m,m-\eta\hat{\jmath}}(u)$ is given by

$$\phi^{(k)}_{m,m-\eta\hat{\jmath}}(u)=\theta^{(k)}(u+2m_j), \tag{2.30}$$

where the functions $\theta^{(j)}(u)$ are given in (2.2). Explicitly,

$$\phi_{m,m-\eta\hat{1}}(u)=\begin{pmatrix}\theta^{(1)}(u+2m_1)\\ \theta^{(2)}(u+2m_1)\end{pmatrix},\qquad \phi_{m,m-\eta\hat{2}}(u)=\begin{pmatrix}\theta^{(1)}(u+2m_2)\\ \theta^{(2)}(u+2m_2)\end{pmatrix}. \tag{2.31}$$

One can prove the following identity [41]

$$\det\begin{vmatrix}\theta^{(1)}(u+2m_1) & \theta^{(1)}(u+2m_2)\\ \theta^{(2)}(u+2m_1) & \theta^{(2)}(u+2m_2)\end{vmatrix}=C(\tau)\,\sigma(u+m_1+m_2-\frac{1}{2})\,\sigma(m_{12}),$$

where $C(\tau)$ is non-vanishing constant which only depends on τ. This implies that the two intertwiner vectors $\phi_{m,m-\eta\hat{\imath}}(u)$ are linearly *independent* for a generic $m\in V$.

Using the intertwiner vectors, one can derive the following face-vertex correspondence relation [37]

$$\begin{aligned}&\overline{R}_{1,2}(u_1-u_2)\phi^1_{m,m-\eta\hat{\imath}}(u_1)\phi^2_{m-\eta\hat{\imath},m-\eta(\hat{\imath}+\hat{\jmath})}(u_2)\\ &\qquad=\sum_{k,l}R(u_1-u_2;m)^{kl}_{ij}\phi^1_{m-\eta\hat{l},m-\eta(\hat{l}+\hat{k})}(u_1)\phi^2_{m,m-\eta\hat{l}}(u_2).\end{aligned} \tag{2.32}$$

Then the QYBE (2.7) of the vertex-type R-matrix $\overline{R}(u)$ is equivalent to the dynamical Yang-Baxter equation (2.24) of the SOS R-matrix $R(u,m)$. For a generic m, we can introduce other types of intertwiners $\bar{\phi}$, $\tilde{\phi}$ which are both row vectors and satisfy the following conditions,

$$\bar{\phi}_{m,m-\eta\hat{\mu}}(u)\,\phi_{m,m-\eta\hat{\nu}}(u)=\delta_{\mu\nu},\quad \tilde{\phi}_{m+\eta\hat{\mu},m}(u)\,\phi_{m+\eta\hat{\nu},m}(u)=\delta_{\mu\nu}, \tag{2.33}$$

from which one can derive the relations,

$$\sum_{\mu=1}^{2}\phi_{m,m-\eta\hat{\mu}}(u)\,\bar{\phi}_{m,m-\eta\hat{\mu}}(u)=\mathrm{id}, \tag{2.34}$$

$$\sum_{\mu=1}^{2}\phi_{m+\eta\hat{\mu},m}(u)\,\tilde{\phi}_{m+\eta\hat{\mu},m}(u)=\mathrm{id}. \tag{2.35}$$

With the help of (2.32)-(2.35), we obtain,

$$\begin{aligned}&\tilde{\phi}^1_{m+\eta\hat{k},m}(u_1)\,\overline{R}_{1,2}(u_1-u_2)\phi^2_{m+\eta\hat{\jmath},m}(u_2)\\&\qquad=\sum_{i,l}R(u_1-u_2;m)^{kl}_{ij}\,\tilde{\phi}^1_{m+\eta(\hat{\imath}+\hat{\jmath}),m+\eta\hat{\jmath}}(u_1)\phi^2_{m+\eta(\hat{k}+\hat{l}),m+\eta\hat{k}}(u_2),\end{aligned} \tag{2.36}$$

$$\begin{aligned}&\tilde{\phi}^1_{m+\eta\hat{k},m}(u_1)\tilde{\phi}^2_{m+\eta(\hat{k}+\hat{l}),m+\eta\hat{k}}(u_2)\,\overline{R}_{1,2}(u_1-u_2)\\&\qquad=\sum_{i,j}R(u_1-u_2;m)^{kl}_{ij}\,\tilde{\phi}^1_{m+\eta(\hat{\imath}+\hat{\jmath}),m+\eta\hat{\jmath}}(u_1)\tilde{\phi}^2_{m+\eta\hat{\jmath},m}(u_2),\end{aligned} \tag{2.37}$$

$$\begin{aligned}&\bar{\phi}^2_{m,m-\eta\hat{l}}(u_2)\,\overline{R}_{1,2}(u_1-u_2)\phi^1_{m,m-\eta\hat{\imath}}(u_1)\\&\qquad=\sum_{k,j}R(u_1-u_2;m)^{kl}_{ij}\,\phi^1_{m-\eta\hat{l},m-\eta(\hat{k}+\hat{l})}(u_1)\bar{\phi}^2_{m-\eta\hat{\imath},m-\eta(\hat{\imath}+\hat{\jmath})}(u_2),\end{aligned} \tag{2.38}$$

$$\begin{aligned}&\bar{\phi}^1_{m-\eta\hat{l},m-\eta(\hat{k}+\hat{l})}(u_1)\bar{\phi}^2_{m,m-\eta\hat{l}}(u_2)\,\overline{R}_{12}(u_1-u_2)\\&\qquad=\sum_{i,j}R(u_1-u_2;m)^{kl}_{ij}\,\bar{\phi}^1_{m,m-\eta\hat{\imath}}(u_1)\bar{\phi}^2_{m-\eta\hat{\imath},m-\eta(\hat{\imath}+\hat{\jmath})}(u_2).\end{aligned} \tag{2.39}$$

In addition to the Riemann identity (2.3), the σ-function enjoys the following properties:

$$\sigma(2u)=\frac{2\sigma(u)\,\sigma_{(0,1)}(u)\,\sigma_{(1,0)}(u)\,\sigma_{(1,1)}(u)}{\sigma_{(0,1)}(0)\,\sigma_{(1,0)}(0)\,\sigma_{(1,1)}(0)}, \tag{2.40}$$

$$\sigma(u+1)=-\sigma(u),\qquad \sigma(u+\tau)=e^{-2i\pi(u+\frac{1}{2}+\frac{\tau}{2})}\sigma(u), \tag{2.41}$$

where the functions $\sigma_a(u)$ are given by (2.4). Using the above identities and the method in [41], after tedious calculations, we can show that the K-matrices $K^{\pm}(u)$ given by (2.13) and (2.15) can be expressed in terms of the intertwiners and *diagonal* matrices $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda|u)$ as follows

$$K^-(u)^s_t=\sum_{i,j}\phi^{(s)}_{\lambda-\eta(\hat{\imath}-\hat{\jmath}),\ \lambda-\eta\hat{\imath}}(u)\mathcal{K}(\lambda|u)^j_i\bar{\phi}^{(t)}_{\lambda,\ \lambda-\eta\hat{\imath}}(-u), \tag{2.42}$$

$$K^+(u)^s_t=\sum_{i,j}\phi^{(s)}_{\lambda,\ \lambda-\eta\hat{\jmath}}(-u)\tilde{\mathcal{K}}(\lambda|u)^j_i\tilde{\phi}^{(t)}_{\lambda-\eta(\hat{\jmath}-\hat{\imath}),\ \lambda-\eta\hat{\jmath}}(u). \tag{2.43}$$

Here the two *diagonal* matrices $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda|u)$ are given by

$$\mathcal{K}(\lambda|u)\equiv\mathrm{Diag}(k(\lambda|u)_1,\,k(\lambda|u)_2)=\mathrm{Diag}(\frac{\sigma(\lambda_1+\xi-u)}{\sigma(\lambda_1+\xi+u)},\,\frac{\sigma(\lambda_2+\xi-u)}{\sigma(\lambda_2+\xi+u)}), \tag{2.44}$$

$$\tilde{\mathcal{K}}(\lambda|u) \equiv \mathrm{Diag}(\tilde{k}(\lambda|u)_1,\, \tilde{k}(\lambda|u)_2)$$
$$= \mathrm{Diag}(\frac{\sigma(\lambda_{12}-\eta)\sigma(\lambda_1+\bar{\xi}+u+\eta)}{\sigma(\lambda_{12})\sigma(\lambda_1+\bar{\xi}-u-\eta)},\, \frac{\sigma(\lambda_{12}+\eta)\sigma(\lambda_2+\bar{\xi}+u+\eta)}{\sigma(\lambda_{12})\sigma(\lambda_2+\bar{\xi}-u-\eta)}). \quad (2.45)$$

Although the vertex type K-matrices $K^{\pm}(u)$ given by (2.13) and (2.15) are generally non-diagonal, after the face-vertex transformations (2.42) and (2.43), the face type counterparts $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda|u)$ become *simultaneously* diagonal. This fact enabled the authors in [16, 30] to diagonalize the transfer matrices $t(u)$ (2.12) by applying the generalized algebraic Bethe ansatz method developed in [15]. .

2.2 Two sets of eigenstates

In order to construct the Bethe states of the open XYZ model with non-diagonal boundary terms specified by the K-matrices (2.14) and (2.16), we need to introduce the new double-row monodromy matrices $\mathcal{T}^{\pm}(m|u)$ [15, 32, 42]:

$$\mathcal{T}^{-}(m|u)^{\nu}_{\mu} = \tilde{\phi}^{0}_{m-\eta(\hat{\mu}-\hat{\nu}),m-\eta\hat{\mu}}(u)\ \mathbb{T}_0(u)\phi^{0}_{m,m-\eta\hat{\mu}}(-u), \quad (2.46)$$
$$\mathcal{T}^{+}(m|u)^{j}_{i} = \prod_{k\neq j}\frac{\sigma(m_{jk})}{\sigma(m_{jk}-\eta)}\,\phi^{t_0}_{m-\eta(\hat{j}-\hat{\imath}),m-\eta\hat{j}}(u)\left(\mathbb{T}^{+}(u)\right)^{t_0}\bar{\phi}^{t_0}_{m,m-\eta\hat{j}}(-u), \quad (2.47)$$

where t_0 denotes transposition in the 0-th space (i.e. auxiliary space) and $\mathbb{T}^{+}(u)$ is given by

$$\left(\mathbb{T}^{+}(u)\right)^{t_0} = T^{t_0}(u)\left(K^{+}(u)\right)^{t_0}\hat{T}^{t_0}(u). \quad (2.48)$$

These double-row monodromy matrices, in the face picture, can be expressed in terms of the face type R-matrix $R(u;m)$ (2.21) and K-matrices $\mathcal{K}(\lambda|u)$ (2.44) and $\tilde{\mathcal{K}}(\lambda|u)$ (2.45) (for the details see section 3).

So far only two sets of Bethe states (i.e. eigenstates) of the transfer matrix for the models with non-diagonal boundary terms have been found [43, 30, 42]. These two sets of states are

$$|\{v^{(1)}_i\}\rangle^{(I)} = \mathcal{T}^{+}(\lambda+2\eta\hat{1}|v^{(1)}_1)^1_2\cdots\mathcal{T}^{+}(\lambda+2M\eta\hat{1}|v^{(1)}_M)^1_2|\Omega^{(I)}(\lambda)\rangle, \quad (2.49)$$
$$|\{v^{(2)}_i\}\rangle^{(II)} = \mathcal{T}^{-}(\lambda-2\eta\hat{2}|v^{(2)}_1)^2_1\cdots\mathcal{T}^{-}(\lambda-2M\eta\hat{2}|v^{(2)}_M)^2_1|\Omega^{(II)}(\lambda)\rangle, \quad (2.50)$$

where the vector λ is related to the boundary parameters (2.17). The associated reference states $|\Omega^{(I)}(\lambda)\rangle$ and $|\Omega^{(II)}(\lambda)\rangle$ are

$$|\Omega^{(I)}(\lambda)\rangle = \phi^{1}_{\lambda+N\eta\hat{1},\lambda+(N-1)\eta\hat{1}}(z_1)\phi^{2}_{\lambda+(N-1)\eta\hat{1},\lambda+(N-2)\eta\hat{1}}(z_2)\cdots\phi^{N}_{\lambda+\eta\hat{1},\lambda}(z_N), \quad (2.51)$$
$$|\Omega^{(II)}(\lambda)\rangle = \phi^{1}_{\lambda,\lambda-\eta\hat{2}}(z_1)\phi^{2}_{\lambda-\eta\hat{2},\lambda-2\eta\hat{2}}(z_2)\cdots\phi^{N}_{\lambda-(N-1)\eta\hat{2},\lambda-N\eta\hat{2}}(z_N). \quad (2.52)$$

It is remarked that $\phi^k = \mathrm{id}\otimes\mathrm{id}\cdots\otimes \overset{k-th}{\phi} \otimes\mathrm{id}\cdots$.

If the parameters $\{v_k^{(1)}\}$ satisfy the first set of Bethe ansatz equations given by

$$\begin{aligned}
&\frac{\sigma(\lambda_2+\xi+v_\alpha^{(1)})\sigma(\lambda_2+\bar{\xi}-v_\alpha^{(1)})\sigma(\lambda_1+\bar{\xi}+v_\alpha^{(1)})\sigma(\lambda_1+\xi-v_\alpha^{(1)})}{\sigma(\lambda_2+\bar{\xi}+v_\alpha^{(1)}+\eta)\sigma(\lambda_2+\xi-v_\alpha^{(1)}-\eta)\sigma(\lambda_1+\xi+v_\alpha^{(1)}+\eta)\sigma(\lambda_1+\bar{\xi}-v_\alpha^{(1)}-\eta)}\\
&\quad=\prod_{k\neq\alpha}^{M}\frac{\sigma(v_\alpha^{(1)}+v_k^{(1)}+2\eta)\sigma(v_\alpha^{(1)}-v_k^{(1)}+\eta)}{\sigma(v_\alpha^{(1)}+v_k^{(1)})\sigma(v_\alpha^{(1)}-v_k^{(1)}-\eta)}\\
&\quad\quad\times\prod_{k=1}^{2M}\frac{\sigma(v_\alpha^{(1)}+z_k)\sigma(v_\alpha^{(1)}-z_k)}{\sigma(v_\alpha^{(1)}+z_k+\eta)\sigma(v_\alpha^{(1)}-z_k+\eta)},\quad \alpha=1,\cdots,M,
\end{aligned}\tag{2.53}$$

the Bethe state $|v_1^{(1)},\cdots,v_M^{(1)}\rangle^{(1)}$ becomes the eigenstate of the transfer matrix with eigenvalue $\Lambda^{(1)}(u)$ given by [42]

$$\begin{aligned}
\Lambda^{(1)}(u) &= \frac{\sigma(\lambda_2+\bar{\xi}-u)\sigma(\lambda_1+\bar{\xi}+u)\sigma(\lambda_1+\xi-u)\sigma(2u+2\eta)}{\sigma(\lambda_2+\bar{\xi}-u-\eta)\sigma(\lambda_1+\bar{\xi}-u-\eta)\sigma(\lambda_1+\xi+u)\sigma(2u+\eta)}\\
&\quad\quad\times\prod_{k=1}^{M}\frac{\sigma(u+v_k^{(1)})\sigma(u-v_k^{(1)}-\eta)}{\sigma(u+v_k^{(1)}+\eta)\sigma(u-v_k^{(1)})}\\
&\quad+\frac{\sigma(\lambda_2+\bar{\xi}+u+\eta)\sigma(\lambda_1+\xi+u+\eta)\sigma(\lambda_2+\xi-u-\eta)\sigma(2u)}{\sigma(\lambda_2+\bar{\xi}-u-\eta)\sigma(\lambda_1+\xi+u)\sigma(\lambda_2+\xi+u)\sigma(2u+\eta)}\\
&\quad\quad\times\prod_{k=1}^{M}\frac{\sigma(u+v_k^{(1)}+2\eta)\sigma(u-v_k^{(1)}+\eta)}{\sigma(u+v_k^{(1)}+\eta)\sigma(u-v_k^{(1)})}\\
&\quad\quad\times\prod_{k=1}^{2M}\frac{\sigma(u+z_k)\sigma(u-z_k)}{\sigma(u+z_k+\eta)\sigma(u-z_k+\eta)}.
\end{aligned}\tag{2.54}$$

If the parameters $\{v_k^{(2)}\}$ satisfy the second Bethe Ansatz equations

$$\begin{aligned}
&\frac{\sigma(\lambda_1+\xi+v_\alpha^{(2)})\sigma(\lambda_1+\bar{\xi}-v_\alpha^{(2)})\sigma(\lambda_2+\bar{\xi}+v_\alpha^{(2)})\sigma(\lambda_2+\xi-v_\alpha^{(2)})}{\sigma(\lambda_1+\bar{\xi}+v_\alpha^{(2)}+\eta)\sigma(\lambda_1+\xi-v_\alpha^{(2)}-\eta)\sigma(\lambda_2+\xi+v_\alpha^{(2)}+\eta)\sigma(\lambda_2+\bar{\xi}-v_\alpha^{(2)}-\eta)}\\
&\quad=\prod_{k\neq\alpha}^{M}\frac{\sigma(v_\alpha^{(2)}+v_k^{(2)}+2\eta)\sigma(v_\alpha^{(2)}-v_k^{(2)}+\eta)}{\sigma(v_\alpha^{(2)}+v_k^{(2)})\sigma(v_\alpha^{(2)}-v_k^{(2)}-\eta)}\\
&\quad\quad\times\prod_{k=1}^{2M}\frac{\sigma(v_\alpha^{(2)}+z_k)\sigma(v_\alpha^{(2)}-z_k)}{\sigma(v_\alpha^{(2)}+z_k+\eta)\sigma(v_\alpha^{(2)}-z_k+\eta)},\quad \alpha=1,\cdots,M,
\end{aligned}\tag{2.55}$$

the Bethe states $|v_1^{(2)},\cdots,v_M^{(2)}\rangle^{(II)}$ yield the second set of the eigenstates of the transfer matrix with the eigenvalues [43, 30],

$$\Lambda^{(2)}(u) = \frac{\sigma(2u+2\eta)\sigma(\lambda_1+\bar{\xi}-u)\sigma(\lambda_2+\bar{\xi}+u)\sigma(\lambda_2+\xi-u)}{\sigma(2u+\eta)\sigma(\lambda_1+\bar{\xi}-u-\eta)\sigma(\lambda_2+\bar{\xi}-u-\eta)\sigma(\lambda_2+\xi+u)}$$

$$
\begin{aligned}
&\times \prod_{k=1}^{M} \frac{\sigma(u+v_k^{(2)})\sigma(u-v_k^{(2)}-\eta)}{\sigma(u+v_k^{(2)}+\eta)\sigma(u-v_k^{(2)})} \\
+&\frac{\sigma(2u)\sigma(\lambda_1+\bar{\xi}+u+\eta)\sigma(\lambda_2+\xi+u+\eta)\sigma(\lambda_1+\xi-u-\eta)}{\sigma(2u+\eta)\sigma(\lambda_1+\bar{\xi}-u-\eta)\sigma(\lambda_2+\xi+u)\sigma(\lambda_1+\xi+u)} \\
&\times \prod_{k=1}^{M} \frac{\sigma(u+v_k^{(2)}+2\eta)\sigma(u-v_k^{(2)}+\eta)}{\sigma(u+v_k^{(2)}+\eta)\sigma(u-v_k^{(2)})} \\
&\times \prod_{k=1}^{2M} \frac{\sigma(u+z_k)\sigma(u-z_k)}{\sigma(u+z_k+\eta)\sigma(u-z_k+\eta)}. \qquad (2.56)
\end{aligned}
$$

3 $\mathcal{T}^{\pm}(m|u)$ in the face picture

The K-matrices $K^{\pm}(u)$ given by (2.13) and (2.15) are generally non-diagonal (in the vertex picture), after the face-vertex transformations (2.42) and (2.43), the face type counterparts $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda|u)$ given by (2.44) and (2.45) *simultaneously* become diagonal. This fact suggests that it would be much simpler if one performs all calculations in the face picture [44].

Let us introduce the face type one-row monodromy matrix (c.f (2.8))

$$
\begin{aligned}
T_F(l|u) &\equiv T^F_{0,1\ldots N}(l|u) \\
&= R_{0,N}(u-z_N;l-\eta\sum_{i=1}^{N-1}h^{(i)})\ldots R_{0,2}(u-z_2;l-\eta h^{(1)})R_{0,1}(u-z_1;l), \\
&= \begin{pmatrix} T_F(l|u)^1_1 & T_F(l|u)^1_2 \\ T_F(l|u)^2_1 & T_F(l|u)^2_2 \end{pmatrix} \qquad (3.1)
\end{aligned}
$$

where l is a generic vector in V. The monodromy matrix satisfies the face type quadratic exchange relation [45, 46]. Applying $T_F(l|u)^i_j$ to an arbitrary vector $|i_1,\ldots,i_N\rangle$ in the N-tensor product space $V^{\otimes N}$ given by

$$
|i_1,\ldots,i_N\rangle = \epsilon^1_{i_1}\ldots\epsilon^N_{i_N}, \qquad (3.2)
$$

we have

$$
\begin{aligned}
T_F(l|u)^i_j|i_1,\ldots,i_N\rangle &\equiv T_F(m;l|u)^i_j|i_1,\ldots,i_N\rangle \\
&= \sum_{\alpha_{N-1}\ldots\alpha_1}\sum_{i'_N\ldots i'_1} R(u-z_N;l-\eta\sum_{k=1}^{N-1}\hat{\imath}'_k)^{i\ \ i'_N}_{\alpha_{N-1}\,i_N}\ldots \\
&\qquad \times R(u-z_2;l-\eta\hat{\imath}'_1)^{\alpha_2\,i'_2}_{\alpha_1\,i_2}R(u-z_1;l)^{\alpha_1\,i'_1}_{j\ \ i_1}\,|i'_1,\ldots,i'_N\rangle, \qquad (3.3)
\end{aligned}
$$

where $m = l - \eta \sum_{k=1}^{N} \hat{\imath}_k$. We shall express the double-row monodromy matrices $\mathcal{T}^{\pm}$ given by (2.46) and (2.47) in terms of the above face-type one-row monodromy matrix.

Associated with the vertex type monodromy matrices $T(u)$ (2.8) and $\hat{T}(u)$ (2.11), we introduce the following operators

$$\begin{aligned} T(m,l|u)_\mu^j &= \tilde{\phi}^0_{m+\eta\hat{\jmath},m}(u)\, T_0(u)\, \phi^0_{l+\eta\hat{\mu},l}(u), & (3.4)\\ S(m,l|u)_i^\mu &= \bar{\phi}^0_{l,l-\eta\hat{\mu}}(-u)\, \hat{T}_0(u)\, \phi^0_{m,m-\eta\hat{\imath}}(-u). & (3.5)\end{aligned}$$

Moreover, for the case of $m = l - \eta \sum_{k=1}^{N} \hat{\imath}_k$, we introduce a generic state in the quantum space from the intertwiner vector (2.30)

$$|i_1,\ldots,i_N\rangle_l^m = \phi^1_{l,l-\eta\hat{\imath}_1}(z_1)\phi^2_{l-\eta\hat{\imath}_1,l-\eta(\hat{\imath}_1+\hat{\imath}_2)}(z_2)\ldots\phi^N_{l-\eta\sum_{k=1}^{N-1}\hat{\imath}_k,l-\eta\sum_{k=1}^{N}\hat{\imath}_k}(z_N). \quad (3.6)$$

We can evaluate the action of the operator $T(m,l|u)$ on the state $|i_1,\ldots,i_N\rangle_l^m$ from the face-vertex correspondence relation (2.32)

$$\begin{aligned} T(m,l|u)_\mu^j|i_1,\ldots,i_N\rangle_l^m &= \tilde{\phi}^0_{m+\eta\hat{\jmath},m}(u)\, T_0(u)\, \phi^0_{l+\eta\hat{\mu},l}(u)|i_1,\ldots,i_N\rangle_l^m\\ &= \tilde{\phi}^0_{m+\eta\hat{\jmath},m}(u)\overline{R}_{0,N}(u-z_N)\ldots\overline{R}_{0,1}(u-z_1)\phi^0_{l+\eta\hat{\mu},l}(u)\phi^1_{l,l-\eta\hat{\imath}_1}(z_1)\ldots\\ &= \sum_{\alpha_1,i'_1} R(u-z_1;l+\eta\hat{\mu})_{\mu\ i_1}^{\alpha_1 i'_1}\phi^1_{l+\eta\hat{\mu},l+\eta\hat{\mu}-\eta\hat{\imath}'_1}(z_1)\tilde{\phi}^0_{m+\eta\hat{\jmath},m}(u)\overline{R}_{0,N}(u-z_N)\ldots\\ &\qquad\times\overline{R}_{0,2}(u-z_2)\phi^0_{l+\eta\hat{\mu}-\eta\hat{\imath}'_1,l-\eta\hat{\imath}_1}(u)\phi^2_{l-\eta\hat{\imath}_1,l-\eta(\hat{\imath}_1+\hat{\imath}_2)}(z_2)\ldots\\ &\vdots\\ &= \sum_{\alpha_1\ldots\alpha_{N-1}}\sum_{i'_1\ldots i'_N} R(u-z_N;l+\eta\hat{\mu}-\eta\sum_{k=1}^{N-1}\hat{\imath}'_k)_{\alpha_{N-1}i_N}^{j\ \ \ \ i'_N}\ldots\\ &\qquad\times R(u-z_1;l+\eta\hat{\mu})_{\mu\ i_1}^{\alpha_1 i'_1}|i'_1,\ldots,i'_N\rangle_{l+\eta\hat{\mu}}^{l+\eta\hat{\mu}-\eta\sum_{k=1}^{N}\hat{\imath}'_k}. \end{aligned} \quad (3.7)$$

Here we have used the following property of the R-matrix

$$R(u;m)_{ij}^{i'j'} = R(u;m\pm\eta(\hat{\imath}+\hat{\jmath}))_{ij}^{i'j'} = R(u;m\pm\eta(\hat{\imath}'+\hat{\jmath}'))_{ij}^{i'j'},$$

and the weight conservation condition (2.26). Comparing with (3.3), we have the following correspondence

$$T(m,l|u)_\mu^j|i_1,\ldots,i_N\rangle_l^m \longleftrightarrow T_F(m+\eta\hat{\mu};l+\eta\hat{\mu}|u)_\mu^j|i_1,\ldots,i_N\rangle, \quad (3.8)$$

where vector $|i_1,\ldots,i_N\rangle$ is given by (3.2). Hereafter, we will use O_F to denote the face version of operator O in the face picture.

Noting that

$$\hat{T}_0(u) = \overline{R}_{1,0}(u+z_1)\ldots\overline{R}_{N,0}(u+z_N),$$

we obtain the action of $S(m,l|u)_i^\mu$ on the state $|i_1,\ldots,i_N\rangle_l^m$

$$S(m,l|u)_i^\mu|i_1,\ldots,i_N\rangle_l^m = \sum_{\alpha_1\ldots\alpha_{N-1}}\sum_{i'_1\ldots i'_N} R(u+z_1;l)_{i_1\alpha_{N-1}}^{i'_1\,\mu} R(u+z_2;l-\eta\hat{\imath}_1)_{i_2\alpha_{N-2}}^{i'_2\,\alpha_{N-1}} \\ \times\ldots R(u+z_N;l-\eta\sum_{k=1}^{N-1}\hat{\imath}_k)_{i_N\,i}^{i'_N\,\alpha_1}|i'_1,\ldots,i'_N\rangle_{l-\eta\hat{\mu}}^{l-\eta\hat{\mu}-\eta\sum_{k=1}^N\hat{\imath}'_k}. \tag{3.9}$$

Then the crossing relation of the R-matrix (2.28) enables us to establish the following relation:

$$S(m,l|u)_i^\mu = \varepsilon_{\bar{i}}\varepsilon_{\bar{\mu}}\frac{\sigma(m_{21})}{\sigma(l_{21})}\prod_{k=1}^N\frac{\sigma(u+z_k)}{\sigma(u+z_k+\eta)}T(m,l|-u-\eta)_{\bar{\mu}}^{\bar{i}}, \tag{3.10}$$

where the parities are defined in (2.29) and m_{21} (or l_{21}) is defined in (2.20).

Now we are in the position to express $\mathcal{T}^\pm$ (2.46) and (2.47) in terms of $T(m,l)_j^i$ and $S(l,m)_j^i$. By (2.34) and (2.35), we have

$$\begin{aligned}
\mathcal{T}^-(m|u)_i^j &= \tilde{\phi}^0_{m-\eta(\hat{\imath}-\hat{\jmath}),m-\eta\hat{\imath}}(u)\ \mathbb{T}(u)\ \phi^0_{m,m-\eta\hat{\imath}}(-u)\\
&= \tilde{\phi}^0_{m-\eta(\hat{\imath}-\hat{\jmath}),m-\eta\hat{\imath}}(u)T_0(u)K_0^-(u)\hat{T}_0(u)\phi^0_{m,m-\eta\hat{\imath}}(-u)\\
&= \sum_{\mu,\nu}\tilde{\phi}^0_{m-\eta(\hat{\imath}-\hat{\jmath}),m-\eta\hat{\imath}}(u)T_0(u)\phi^0_{l-\eta(\hat{\nu}-\hat{\mu}),l-\eta\hat{\nu}}(u)\tilde{\phi}^0_{l-\eta(\hat{\nu}-\hat{\mu}),l-\eta\hat{\nu}}(u)\\
&\qquad\times K_0^-(u)\phi^0_{l,l-\eta\hat{\nu}}(-u)\bar{\phi}^0_{l,l-\eta\hat{\nu}}(-u)\hat{T}_0(u)\phi^0_{m,m-\eta\hat{\imath}}(-u)\\
&= \sum_{\mu,\nu}T(m-\eta\hat{\imath},l-\eta\hat{\nu}|u)_\mu^j\mathcal{K}(l|u)_\nu^\mu S(m,l|u)_i^\nu\\
&\stackrel{\rm def}{=} \mathcal{T}^-(m,l|u)_i^j,
\end{aligned} \tag{3.11}$$

where the face-type K-matrix $\mathcal{K}(l|u)_\nu^\mu$ is given by

$$\mathcal{K}(l|u)_\nu^\mu = \tilde{\phi}^0_{l-\eta(\hat{\nu}-\hat{\mu}),l-\eta\hat{\nu}}(u)K_0^-(u)\phi^0_{l,l-\eta\hat{\nu}}(-u). \tag{3.12}$$

Similarly, we have

$$\begin{aligned}
\mathcal{T}^+(m|u)_i^j &= \prod_{k\neq j}\frac{\sigma(m_{jk})}{\sigma(m_{jk}-\eta)}\sum_{\mu,\nu}T(l-\eta\hat{\mu},m-\eta\hat{\jmath}|u)_i^\nu\tilde{\mathcal{K}}(l|u)_\nu^\mu S(l,m|u)_\mu^j\\
&\stackrel{\rm def}{=} \mathcal{T}^+(l,m|u)_i^j
\end{aligned} \tag{3.13}$$

with

$$\tilde{\mathcal{K}}(l|u)^{\mu}_{\nu}=\bar{\phi}^{0}_{l,l-\eta\hat{\mu}}(-u)K^{+}_{0}(u)\phi^{0}_{l-\eta(\hat{\mu}-\hat{\nu}),l-\eta\hat{\mu}}(u). \tag{3.14}$$

Thanks to the fact that when $l=\lambda$ the corresponding face-type K-matrices $\mathcal{K}(\lambda|u)$ (3.12) and $\tilde{\mathcal{K}}(\lambda|u)$ (3.14) become diagonal ones (2.44) and (2.45), we have

$$\mathcal{T}^{-}(m,\lambda|u)^{j}_{i}=\sum_{\mu}T(m-\eta\hat{\imath},\lambda-\eta\hat{\mu}|u)^{j}_{\mu}k(\lambda|u)_{\mu}S(m,\lambda|u)^{\mu}_{i}, \tag{3.15}$$

$$\mathcal{T}^{+}(\lambda,m|u)^{j}_{i}=\prod_{k\neq j}\frac{\sigma(m_{jk})}{\sigma\left(m_{jk}-\eta\right)}\sum_{\mu}T(\lambda-\eta\hat{\mu},m-\eta\hat{\jmath}|u)^{\mu}_{i}\tilde{k}(\lambda|u)_{\mu}S(\lambda,m|u)^{j}_{\mu}, \tag{3.16}$$

where the functions $k(\lambda|u)_{\mu}$ and $\tilde{k}(\lambda|u)_{\mu}$ are given by (2.44) and (2.45) respectively. The relation (3.10) implies that one can further express $\mathcal{T}^{\pm}(m|u)^{j}_{i}$ in terms of only $T(m,l|u)^{j}_{i}$. Here we present the results for the pseudo-particle creation operators $\mathcal{T}^{-}(m|u)^{2}_{1}$ in (2.50) and $\mathcal{T}^{+}(m|u)^{1}_{2}$ in (2.49):

$$\begin{aligned}\mathcal{T}^{-}(m|u)^{2}_{1}=\mathcal{T}^{-}(m,\lambda|u)^{2}_{1}=&\frac{\sigma(m_{21})}{\sigma(\lambda_{21})}\prod_{k=1}^{N}\frac{\sigma(u+z_k)}{\sigma(u+z_k+\eta)}\\&\times\left\{\frac{\sigma(\lambda_1+\xi-u)}{\sigma(\lambda_1+\xi+u)}T(m+\eta\hat{2},\lambda+\eta\hat{2}|u)^{2}_{1}T(m,\lambda|-u-\eta)^{2}_{2}\right.\\&\left.-\frac{\sigma(\lambda_2+\xi-u)}{\sigma(\lambda_2+\xi+u)}T(m+\eta\hat{2},\lambda+\eta\hat{1}|u)^{2}_{2}T(m,\lambda|-u-\eta)^{2}_{1}\right\},\end{aligned} \tag{3.17}$$

$$\begin{aligned}\mathcal{T}^{+}(m|u)^{1}_{2}=\mathcal{T}^{+}(\lambda,m|u)^{1}_{2}=&\prod_{k=1}^{N}\frac{\sigma(u+z_k)}{\sigma(u+z_k+\eta)}\\&\times\left\{\frac{\sigma(\lambda_{12}-\eta)\sigma(\lambda_1+\bar{\xi}+u+\eta)}{\sigma(m_{12}-\eta)\sigma(\lambda_1+\bar{\xi}-u-\eta)}T(\lambda+\eta\hat{2},m+\eta\hat{2}|u)^{1}_{2}T(\lambda,m|-u-\eta)^{2}_{2}\right.\\&\left.-\frac{\sigma(\lambda_{21}-\eta)\sigma(\lambda_2+\bar{\xi}+u+\eta)}{\sigma(m_{21}+\eta)\sigma(\lambda_2+\bar{\xi}-u-\eta)}T(\lambda+\eta\hat{1},m+\eta\hat{2}|u)^{2}_{2}T(\lambda,m|-u-\eta)^{1}_{2}\right\}.\end{aligned} \tag{3.18}$$

Similar to (3.8), we have the correspondence,

$$\mathcal{T}^{-}(m,l|u)^{2}_{1}|i_1,\ldots,i_N\rangle^{m}_{l}\longleftrightarrow\mathcal{T}^{-}_{F}(m,l|u)^{2}_{1}|i_1,\ldots,i_N\rangle, \tag{3.19}$$

$$\mathcal{T}^{+}(m,l|u)^{1}_{2}|i_1,\ldots,i_N\rangle^{m}_{l}\longleftrightarrow\mathcal{T}^{+}_{F}(m,l|u)^{1}_{2}|i_1,\ldots,i_N\rangle. \tag{3.20}$$

It follows from (3.17) and (3.18) that the face-type double-row monodromy matrix elements $\mathcal{T}^{-}_{F}(m,\lambda|u)^{2}_{1}$ and $\mathcal{T}^{+}_{F}(\lambda,m|u)^{1}_{2}$ can be expressed in terms of the face-type one-row monodromy

matrix elements $T_F(m,l|u)^i_j$ (3.3) by

$$\begin{aligned}
\mathcal{T}^-_F(m,\lambda|u)^2_1 &= \frac{\sigma(m_{21})}{\sigma(\lambda_{21})}\prod_{k=1}^{N}\frac{\sigma(u+z_k)}{\sigma(u+z_k+\eta)} \\
&\quad\times\left\{\frac{\sigma(\lambda_1+\xi-u)}{\sigma(\lambda_1+\xi+u)}T_F(m,\lambda|u)^2_1T_F(m+\eta\hat{2},\lambda+\eta\hat{2}|-u-\eta)^2_2\right. \\
&\quad\left.-\frac{\sigma(\lambda_2+\xi-u)}{\sigma(\lambda_2+\xi+u)}T_F(m+2\eta\hat{2},\lambda|u)^2_2T_F(m+\eta\hat{1},\lambda+\eta\hat{1}|-u-\eta)^2_1\right\}, \qquad (3.21)\\
\mathcal{T}^+_F(\lambda,m|u)^1_2 &= \prod_{k=1}^{N}\frac{\sigma(u+z_k)}{\sigma(u+z_k+\eta)} \\
&\quad\times\left\{\frac{\sigma(\lambda_{12}-\eta)\sigma(\lambda_1+\bar{\xi}+u+\eta)}{\sigma(m_{12}-\eta)\sigma(\lambda_1+\bar{\xi}-u-\eta)}T_F(\lambda+2\eta\hat{2},m+2\eta\hat{2}|u)^1_2T_F(\lambda+\eta\hat{2},m+\eta\hat{2}|-u-\eta)^2_2\right. \\
&\quad\left.-\frac{\sigma(\lambda_{21}-\eta)\sigma(\lambda_2+\bar{\xi}+u+\eta)}{\sigma(m_{21}+\eta)\sigma(\lambda_2+\bar{\xi}-u-\eta)}T_F(\lambda,m+2\eta\hat{2}|u)^2_2T_F(\lambda+\eta\hat{2},m+\eta\hat{2}|-u-\eta)^1_2\right\}. \\
&\qquad (3.22)
\end{aligned}$$

In the derivation of the above equations we have used the identity $\hat{1}+\hat{2}=0$. Finally, we obtain the face versions $|v^{(1)}_1,\cdots,v^{(1)}_M\rangle^{(I)}_F$ and $|v^{(1)}_1,\cdots,v^{(1)}_M\rangle^{(II)}_F$ of the two sets of Bethe states (2.49) and (2.50),

$$|v^{(1)}_1,\cdots,v^{(1)}_M\rangle^{(I)}_F = \mathcal{T}^+_F(\lambda,\lambda+2\eta\hat{1}|v^{(1)}_1)^1_2\cdots\mathcal{T}^+_F(\lambda,\lambda+2M\eta\hat{1}|v^{(1)}_M)^1_2|1,\ldots,1\rangle, \quad (3.23)$$

$$|v^{(2)}_1,\cdots,v^{(2)}_M\rangle^{(II)}_F = \mathcal{T}^-_F(\lambda-2\eta\hat{2},\lambda|v^{(2)}_1)^2_1\cdots\mathcal{T}^-_F(\lambda-2M\eta\hat{2},\lambda|v^{(2)}_M)^2_1|2,\ldots,2\rangle. \quad (3.24)$$

In the next section with the help of the Drinfeld twist (or factorizing F-matrix) in the face picture of the eight-vertex SOS model [6], we study expressions of the pseudo-particle creation operators $\mathcal{T}^\pm_F$ given by (3.21) and (3.22) in the F-basis. We find that in the magic F-basis these operators indeed take completely symmetric and polarization free forms simultaneously.

4 F-basis

In this section, after briefly reviewing the result [6] about the Drinfeld twist [3] (factorizing F-matrix) of the eight-vertex SOS model, we obtain the explicit expression of the double rows monodromy operator $\mathcal{T}^\mp_F(m,\lambda|u)^2_1$ given by (3.21) and (3.22) in the F-basis provided by the F-matrix.

4.1 Factorizing Drinfeld twist F

Let $\mathcal{S}_N$ be the permutation group over indices $1,\ldots,N$ and $\{s_i|i=1,\ldots,N-1\}$ be the set of elementary permutations in $\mathcal{S}_N$. For each elementary permutation s_i, we introduce the associated operator $R^{s_i}_{1\ldots N}$ on the quantum space

$$R^{s_i}_{1\ldots N}(l) \equiv R^{s_i}(l) = R_{i,i+1}(z_i - z_{i+1}|l - \eta\sum_{k=1}^{i-1} h^{(k)}), \tag{4.1}$$

where l is a generic vector in V. For any s, $s' \in \mathcal{S}_N$, operator $R^{ss'}_{1\ldots N}$ associated with ss' satisfies the following composition law [2, 6, 8]:

$$R^{ss'}_{1\ldots N}(l) = R^{s'}_{s(1\ldots N)}(l)\, R^{s}_{1\ldots N}(l). \tag{4.2}$$

Let s be decomposed in a minimal way in terms of elementary permutations,

$$s = s_{\beta_1}\ldots s_{\beta_p}, \tag{4.3}$$

where $\beta_i = 1,\ldots,N-1$ and the positive integer p is the length of s. The composition law (4.2) enables one to obtain operator $R^s_{1\ldots N}$ associated with each $s \in \mathcal{S}_N$. The dynamical quantum Yang-Baxter equation (2.24), the weight conservation condition (2.26) and the unitary condition (2.27) guarantee the uniqueness of $R^s_{1\ldots N}$. Moreover, one may check that $R^s_{1\ldots N}$ satisfies the following exchange relation with the face type one-row monodromy matrix (3.1)

$$R^s_{1\ldots N}(l)T^F_{0,1\ldots N}(l|u) = T^F_{0,s(1\ldots N)}(l|u)R^s_{1\ldots N}(l-\eta h^{(0)}), \qquad \forall s \in \mathcal{S}_N. \tag{4.4}$$

Now, we construct the face-type Drinfeld twist $F_{1\ldots N}(l) \equiv F_{1\ldots N}(l;z_1,\ldots,z_N)$ [2] on the N-fold tensor product space $V^{\otimes N}$, which satisfies the following three properties [6, 8]:

$$\text{I. lower} - \text{triangularity}; \tag{4.5}$$

$$\text{II. non} - \text{degeneracy}; \tag{4.6}$$

$$\text{III. factorizing property}: \; R^s_{1\ldots N}(l) = F^{-1}_{s(1\ldots N)}(l)F_{1\ldots N}(l), \; \forall s \in \mathcal{S}_N. \tag{4.7}$$

Substituting (4.7) into the exchange relation (4.4), we have

$$F^{-1}_{s(1\ldots N)}(l)F_{1\ldots N}(l)T^F_{0,1\ldots N}(l|u) = T^F_{0,s(1\ldots N)}(l|u)F^{-1}_{s(1\ldots N)}(l-\eta h^{(0)})F_{1\ldots N}(l-\eta h^{(0)}). \tag{4.8}$$

[2]In this paper, we adopt the convention: $F_{s(1\ldots N)}(l) \equiv F_{s(1\ldots N)}(l;z_{s(1)},\ldots,z_{s(N)})$.

Equivalently,

$$F_{1\dots N}(l)T^F_{0,1\dots N}(l|u)F^{-1}_{1\dots N}(l-\eta h^{(0)})=F_{s(1\dots N)}(l)T^F_{0,s(1\dots N)}(l|u)F^{-1}_{s(1\dots N)}(l-\eta h^{(0)}). \tag{4.9}$$

Let us introduce the twisted monodromy matrix $\tilde{T}^F_{0,1\dots N}(l|u)$ by

$$\begin{aligned}\tilde{T}^F_{0,1\dots N}(l|u) &= F_{1\dots N}(l)T^F_{0,1\dots N}(l|u)F^{-1}_{1\dots N}(l-\eta h^{(0)})\\ &= \begin{pmatrix}\tilde{T}_F(l|u)^1_1 & \tilde{T}_F(l|u)^1_2\\ \tilde{T}_F(l|u)^2_1 & \tilde{T}_F(l|u)^2_2\end{pmatrix}.\end{aligned} \tag{4.10}$$

Then (4.9) implies that the twisted monodromy matrix is symmetric under $\mathcal{S}_N$, namely,

$$\tilde{T}^F_{0,1\dots N}(l|u)=\tilde{T}^F_{0,s(1\dots N)}(l|u),\quad \forall s\in\mathcal{S}_N. \tag{4.11}$$

Define the F-matrix:

$$F_{1\dots N}(l)=\sum_{s\in\mathcal{S}_N}\sum_{\{\alpha_j\}=1}^{2}{}^{*}\ \prod_{i=1}^{N}P^{s(i)}_{\alpha_{s(i)}}\,R^s_{1\dots N}(l), \tag{4.12}$$

where P^i_α is the embedding of the project operator P_α in the i^{th} space with matric elements $(P_\alpha)_{kl}=\delta_{kl}\delta_{k\alpha}$. The sum $\sum^*$ in (4.12) is over all non-decreasing sequences of the labels $\alpha_{s(i)}$:

$$\alpha_{s(i+1)}\geq\alpha_{s(i)}\quad\text{if}\quad s(i+1)>s(i), \tag{4.13}$$

$$\alpha_{s(i+1)}>\alpha_{s(i)}\quad\text{if}\quad s(i+1)<s(i). \tag{4.14}$$

From (4.14), $F_{1\dots N}(l)$ obviously is a lower-triangular matrix. Moreover, the F-matrix is non-degenerate because all its diagonal elements are non-zero. It was shown in [6] that the F-matrix also satisfies the factorizing property (4.7).

4.2 Completely symmetric representations

In the F-basis provided by the F-matrix (4.12), the twisted operators $\tilde{T}_F(l|u)^j_i$ defined by (4.10) become polarization free [6]. Here we present the results relevant for our purpose

$$\tilde{T}_F(l|u)^2_2=\frac{\sigma(l_{21}-\eta)}{\sigma\left(l_{21}-\eta+\eta\langle H,\epsilon_1\rangle\right)}\otimes_i\begin{pmatrix}\frac{\sigma(u-z_i)}{\sigma(u-z_i+\eta)} & \\ & 1\end{pmatrix}_{(i)}, \tag{4.15}$$

$$\tilde{T}_F(l|u)^2_1=\sum_{i=1}^{N}\frac{\sigma(\eta)\sigma(u-z_i+l_{12})}{\sigma(u-z_i+\eta)\sigma(l_{12})}E^i_{12}\otimes_{j\neq i}\begin{pmatrix}\frac{\sigma(u-z_j)\sigma(z_i-z_j+\eta)}{\sigma(u-z_j+\eta)\sigma(z_i-z_j)} & \\ & 1\end{pmatrix}_{(j)}, \tag{4.16}$$

$$\tilde{T}_F(l|u)^1_2 = \frac{\sigma(l_{21}-\eta)}{\sigma(l_{21}+\eta\langle H,\epsilon_1-\epsilon_2\rangle)}\sum_{i=1}^{N}\frac{\sigma(\eta)\sigma(u-z_i+l_{21}+\eta+\eta\langle H,\epsilon_1-\epsilon_2\rangle)}{\sigma(u-z_i+\eta)\sigma(l_{21}+\eta+\eta\langle H,\epsilon_1-\epsilon_2\rangle)}$$

$$\times E^i_{21}\otimes_{j\neq i}\begin{pmatrix}\frac{\sigma(u-z_j)}{\sigma(u-z_j+\eta)} & \\ & \frac{\sigma(z_j-z_i+\eta)}{\sigma(z_j-z_i)}\end{pmatrix}_{(j)}, \tag{4.17}$$

where $H=\sum_{k=1}^{N}h^{(k)}$. Applying the above operators to the arbitrary state $|i_1,\ldots,i_N\rangle$ given by (3.2), we have

$$\tilde{T}_F(m,l|u)^2_2 = \frac{\sigma(l_{21}-\eta)}{\sigma\left(l_2-m_1-\eta\right)}\otimes_i\begin{pmatrix}\frac{\sigma(u-z_i)}{\sigma(u-z_i+\eta)} & \\ & 1\end{pmatrix}_{(i)}, \tag{4.18}$$

$$\tilde{T}_F(m,l|u)^2_1 = \sum_{i=1}^{N}\frac{\sigma(\eta)\sigma(u-z_i+l_{12})}{\sigma(u-z_i+\eta)\sigma(l_{12})}$$

$$\times E^i_{12}\otimes_{j\neq i}\begin{pmatrix}\frac{\sigma(u-z_j)\sigma(z_i-z_j+\eta)}{\sigma(u-z_j+\eta)\sigma(z_i-z_j)} & \\ & 1\end{pmatrix}_{(j)}, \tag{4.19}$$

$$\tilde{T}_F(m,l|u)^1_2 = \frac{\sigma(l_{21}-\eta)}{\sigma(m_{21}-2\eta)}\sum_{i=1}^{N}\frac{\sigma(\eta)\sigma(u-z_i+m_{21}-\eta)}{\sigma(u-z_i+\eta)\sigma(m_{21}-\eta)}$$

$$\times E^i_{21}\otimes_{j\neq i}\begin{pmatrix}\frac{\sigma(u-z_j)}{\sigma(u-z_j+\eta)} & \\ & \frac{\sigma(z_j-z_i+\eta)}{\sigma(z_j-z_i)}\end{pmatrix}_{(j)}. \tag{4.20}$$

With the help of the Riemann identity (2.3), we find that the two pseudo-particle creation operators (3.21) and (3.22) in the F-basis simultaneously have the following completely symmetric polarization free forms:

$$\tilde{\mathcal{T}}^-_F(m,\lambda|u)^2_1 = \frac{\sigma(m_{12})}{\sigma(m_1-\lambda_2)}\prod_{k=1}^{N}\frac{\sigma(u+z_k)}{\sigma(u+z_k+\eta)}$$

$$\times\sum_{i=1}^{N}\frac{\sigma(\lambda_1+\xi-z_i)\sigma(\lambda_2+\xi+z_i)\sigma(2u)\sigma(\eta)}{\sigma(\lambda_1+\xi+u)\sigma(\lambda_2+\xi+u)\sigma(u-z_i+\eta)\sigma(u+z_i)}$$

$$\times E^i_{12}\otimes_{j\neq i}\begin{pmatrix}\frac{\sigma(u-z_j)\sigma(u+z_j+\eta)\sigma(z_i-z_j+\eta)}{\sigma(u-z_j+\eta)\sigma(u+z_j)\sigma(z_i-z_j)} & \\ & 1\end{pmatrix}_{(j)}, \tag{4.21}$$

$$\tilde{\mathcal{T}}^+_F(\lambda,m|u)^1_2 = \frac{\sigma(m_{21}+\eta)}{\sigma(m_2-\lambda_1)}\prod_{k=1}^{N}\frac{\sigma(u+z_k)}{\sigma(u+z_k+\eta)}$$

$$\times\sum_{i=1}^{N}\frac{\sigma(\lambda_2+\bar{\xi}-z_i)\sigma(\lambda_1+\bar{\xi}+z_i)\sigma(2u+2\eta)\sigma(\eta)}{\sigma(\lambda_1+\bar{\xi}-u-\eta)\sigma(\lambda_2+\bar{\xi}-u-\eta)\sigma(u+z_i)\sigma(u-z_i+\eta)}$$

$$\times E^i_{21}\otimes_{j\neq i}\begin{pmatrix}\frac{\sigma(u-z_j)\sigma(u+z_j+\eta)}{\sigma(u-z_j+\eta)\sigma(u+z_j)} & \\ & \frac{\sigma(z_j-z_i+\eta)}{\sigma(z_j-z_i)}\end{pmatrix}_{(j)}. \tag{4.22}$$

The very polarization free form (4.21) of $\tilde{\mathcal{T}}^-_F(m,\lambda|u)^2_1$ enabled the authors in [44] to succeed in obtaining a single determinant representation of the domain wall partition function of the eight-vertex model with a non-diagonal reflection end.

5 Bethe states in F-basis

Now let us evaluate the two sets of Bethe states (3.23) and (3.24) in the F-basis (or the twisted Bethe states)

$$\overline{|v^{(1)}_1,\cdots,v^{(1)}_M\rangle_F}^{(I)} = F_{1\ldots N}(\lambda)\,|v^{(1)}_1,\cdots,v^{(1)}_M\rangle^{(I)}_F, \tag{5.1}$$

$$\overline{|v^{(2)}_1,\cdots,v^{(2)}_M\rangle_F}^{(II)} = F_{1\ldots N}(\lambda)\,|v^{(2)}_1,\cdots,v^{(2)}_M\rangle^{(II)}_F. \tag{5.2}$$

Since $|1,\ldots,1\rangle$ and $|2,\ldots,2\rangle$ are invariant under the action of the F-matrix $F_{1\ldots N}(l)$ (4.12), namely,

$$F_{1\ldots N}(l)\,|i,\ldots,i\rangle=|i,\ldots,i\rangle,\quad i=1,2,$$

we have

$$\overline{|v^{(1)}_1,\cdots,v^{(1)}_M\rangle_F}^{(I)}=\tilde{\mathcal{T}}^+_F(\lambda,\lambda+2\eta\hat{1}|v^{(1)}_1)^1_2\cdots\tilde{\mathcal{T}}^+_F(\lambda,\lambda+2M\eta\hat{1}|v^{(1)}_M)^1_2|1,\ldots,1\rangle, \tag{5.3}$$

$$\overline{|v^{(2)}_1,\cdots,v^{(2)}_M\rangle_F}^{(II)}=\tilde{\mathcal{T}}^-_F(\lambda-2\eta\hat{2},\lambda|v^{(2)}_1)^2_1\cdots\tilde{\mathcal{T}}^-_F(\lambda-2M\eta\hat{2},\lambda|v^{(2)}_M)^2_1|2,\ldots,2\rangle. \tag{5.4}$$

Thanks to the polarization free representations (4.21) and (4.22) of the pseudo-particle creation operators, we can obtain completely symmetric expressions of the two sets of Bethe sates in the F-basis:

$$\begin{aligned}\overline{|v^{(1)}_1,\cdots,v^{(1)}_M\rangle_F}^{(I)} &= \prod_{k=1}^M\left\{\frac{\sigma(\lambda_{12}-\eta+2k\eta)}{\sigma(\lambda_{12}+k\eta)}\prod_{n=1}^N\frac{\sigma(v^{(1)}_k-z_n)}{\sigma(v^{(1)}_k-z_n+\eta)}\right\}\\ &\quad\times\sum_{i_1<i_2\ldots<i_M}B^{(I)}_M\left(\{v^{(1)}_\alpha\}|\{z_{i_n}\}\right)E^{i_1}_{21}\ldots E^{i_M}_{21}\,|1,\ldots,1\rangle,\end{aligned} \tag{5.5}$$

$$\begin{aligned}\overline{|v^{(2)}_1,\cdots,v^{(2)}_M\rangle_F}^{(II)} &= \prod_{k=1}^M\left\{\frac{\sigma(\lambda_{12}+2k\eta)}{\sigma(\lambda_{12}+k\eta)}\prod_{n=1}^N\frac{\sigma(v^{(2)}_k+z_n)}{\sigma(v^{(2)}_k+z_n+\eta)}\right\}\\ &\quad\times\sum_{i_1<i_2\ldots<i_M}B^{(II)}_M\left(\{v^{(2)}_\alpha\}|\{z_{i_n}\}\right)E^{i_1}_{12}\ldots E^{i_M}_{12}\,|2,\ldots,2\rangle.\end{aligned} \tag{5.6}$$

Here the functions $B_M^{(I)}(\{v_\alpha\}|\{z_i\})$ and $B_M^{(II)}(\{v_\alpha\}|\{z_i\})$ are given by

$$
\begin{aligned}
B_M^{(I)}(\{v_\alpha\}|\{z_i\}) =& \prod_{n=1}^{M}\prod_{i=1}^{M}\frac{\sigma(v_n-z_i+\eta)\sigma(v_n+z_i)}{\sigma(v_n-z_i)\sigma(v_n+z_i+\eta)} \\
&\times\sum_{s\in\mathcal{S}_M}\prod_{n=1}^{M}\Big\{\frac{\sigma(\lambda_2+\bar{\xi}-z_{s(n)})\sigma(\lambda_1+\bar{\xi}+z_{s(n)})\sigma(2v_n+2\eta)\sigma(\eta)}{\sigma(\lambda_2+\bar{\xi}-v_n-\eta)\sigma(\lambda_1+\bar{\xi}-v_n-\eta)\sigma(v_n+z_{s(n)})\sigma(v_n-z_{s(n)}+\eta)} \\
&\times\prod_{k>n}^{M}\frac{\sigma(v_k-z_{s(n)})\sigma(v_k+z_{s(n)}+\eta)\sigma(z_{s(k)}-z_{s(n)}+\eta)}{\sigma(v_k-z_{s(n)}+\eta)\sigma(v_k+z_{s(n)})\sigma(z_{s(k)}-z_{s(n)})}\Big\},
\end{aligned}
\tag{5.7}
$$

$$
\begin{aligned}
B_M^{(II)}(\{v_\alpha\}|\{z_i\}) =& \sum_{s\in\mathcal{S}_M}\prod_{n=1}^{M}\Big\{\frac{\sigma(\lambda_1+\xi-z_{s(n)})\sigma(\lambda_2+\xi+z_{s(n)})\sigma(2v_n)\sigma(\eta)}{\sigma(\lambda_1+\xi+v_n)\sigma(\lambda_2+\xi+v_n)\sigma(v_n-z_{s(n)}+\eta)\sigma(v_n+z_{s(n)})} \\
&\times\prod_{k>n}^{M}\frac{\sigma(v_n-z_{s(k)})\sigma(v_n+z_{s(k)}+\eta)\sigma(z_{s(n)}-z_{s(k)}+\eta)}{\sigma(v_n-z_{s(k)}+\eta)\sigma(v_n+z_{s(k)})\sigma(z_{s(n)}-z_{s(k)})}\Big\}.
\end{aligned}
\tag{5.8}
$$

From the expressions (5.7) and (5.8), it is easy to check that these two functions $B_M^{(i)}(\{v_\alpha\}|\{z_i\})$ are symmetric functions of $\{v_a\}$ and $\{z_i\}$ separatively. Following the method of [34, 44], we can further express the functions in terms of some single determinant respectively as follows:

$$
B_M^{(I)}(\{v_\alpha\}|\{z_i\}) = \frac{\prod_{\alpha=1}^{M}\prod_{i=1}^{M}\sigma(v_\alpha+z_i)\sigma(v_\alpha-z_i+\eta)\,\det\mathcal{N}^{(I)}(\{v_\alpha\};\{z_i\})}{\prod_{\alpha>\beta}\sigma(v_\alpha-v_\beta)\sigma(v_\alpha+v_\beta+\eta)\prod_{k<l}\sigma(z_l-z_k)\sigma(z_l+z_k)}, \tag{5.9}
$$

$$
B_M^{(II)}(\{v_\alpha\}|\{z_i\}) = \frac{\prod_{\alpha=1}^{M}\prod_{i=1}^{M}\sigma(v_\alpha-z_i)\sigma(v_\alpha+z_i+\eta)\,\det\mathcal{N}^{(II)}(\{v_\alpha\};\{z_i\})}{\prod_{\alpha>\beta}\sigma(v_\alpha-v_\beta)\sigma(v_\alpha+v_\beta+\eta)\prod_{k<l}\sigma(z_k-z_l)\sigma(z_k+z_l)}, \tag{5.10}
$$

where the $M\times M$ matrices $\mathcal{N}^{(i)}(\{v_\alpha\};\{z_i\})$ are given by

$$
\begin{aligned}
\mathcal{N}^{(I)}(\{v_\alpha\};\{z_i\})_{\alpha,j} =& \frac{\sigma(\eta)\sigma(\lambda_2+\bar{\xi}-z_j)}{\sigma(v_\alpha-z_j)\sigma(v_\alpha+z_j+\eta)\sigma(\lambda_2+\bar{\xi}-v_\alpha-\eta)} \\
&\times\frac{\sigma(\lambda_1+\bar{\xi}+z_j)\sigma(2v_\alpha+2\eta)}{\sigma(\lambda_1+\bar{\xi}-v_\alpha-\eta)\sigma(v_\alpha-z_j+\eta)\sigma(v_\alpha+z_j)},
\end{aligned}
\tag{5.11}
$$

$$
\begin{aligned}
\mathcal{N}^{(II)}(\{v_\alpha\};\{z_i\})_{\alpha,j} =& \frac{\sigma(\eta)\sigma(\lambda_1+\xi-z_j)}{\sigma(v_\alpha-z_j)\sigma(v_\alpha+z_j+\eta)\sigma(\lambda_1+\xi+v_\alpha)} \\
&\times\frac{\sigma(\lambda_2+\xi+z_j)\sigma(2v_\alpha)}{\sigma(\lambda_2+\xi+v_\alpha)\sigma(v_\alpha-z_j+\eta)\sigma(v_\alpha+z_j)}.
\end{aligned}
\tag{5.12}
$$

We remark that if the parameters $\{v_k^{(1)}\}$ (or $\{v_k^{(2)}\}$) do not satisfy the associated Bethe ansatz equations (2.53)(or (2.55)), the corresponding twisted states (5.3) and (5.4) become off-shell Bethe states. These off-shell Bethe states can still be expressed in the same forms as those of (5.5)-(5.12) (but the corresponding parameters are not necessarily the roots of the Bethe ansatz equations).

6 Conclusions

We have studied the explicit expressions of the Bethe states of the open XYZ chain with non-diagonal boundary terms (where the non-diagonal K-matrices $K^{\pm}(u)$ are given by (2.14) and (2.16)) by using the Drinfeld twist or factorizing F-matrix for the eight-vertex SOS model. It is found that in the F-basis the pseudo-particle creation operators, which generate the two sets of the eigenstates of the model, simultaneously take the completely symmetric and polarization free forms (4.21) and (4.22). This allows us to obtain the explicit and completely symmetric explicit expressions (5.5) and (5.6) of the two sets of (off-shell) Bethe states. Moreover, the coefficients $B_M^{(i)}(\{v_\alpha\}|\{z_i\})$ in these expressions can be expressed in terms of a single determinant (5.9)-(5.12) respectively. Such a single determinant representation makes it feasible to derive the determinant representations of scalar products of the Bethe states for the open XYZ chain with non-diagonal boundary terms specified by the non-diagonal K-matrices $K^{\pm}(u)$ (2.14) and (2.16).

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